Mathematics for Computer Science: Homework 1

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LPV 1.8.26

Find the number of all 20-digits integers in which no two consecutive digits are the same.

Answer: Let the number be $\overline{a_{20}a_{19}\dots a_1}$, $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then a_{20} can be chosen from $S - \{0\}$, and a_i can be chosen from $S - \{a_{i+1}\}$. So there's 9^{20} such integers.

LPV 1.8.29

What is the number of ways to color n objects with 3 colors if every color must be used at least once?

Answer: According to the Inclusion-Exclusion Formula, the answer is $3^n - \binom{3}{2}2^n + \binom{3}{1}1^n$.

LPV 2.1.8

Prove that $S_n = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Answer: The statement is rather trivial when $n = 1$. Assume that it is true for $n = k$, then we have

$$S_{k+1} = S_k + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)\left(2k^2+7k+6\right)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

It is true for $n = k + 1$. So the statement is true for all positive integer.

LPV 2.1.13

Read carefully the induction proof given by the book. The assertion we proved is clearly wrong: Where is the error?

Answer: When we proving the assertion with $n = 3$, we cannot find d s.t. $d \neq c$, to prove c go through $a \cap b$. So this proof failed for all $n \geq 3$.

LPV 2.5.2

$$\sum_{i=0}^{n} i \binom{n}{i}$$

Answer: By combinatorial arguments,

$$\sum_{i=0}^{n} i \binom{n}{i} = \sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=1}^{n} i \frac{n!}{i!\,(n-i)!} = n \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!\,(n-i)!} = n \sum_{i=1}^{n} \binom{n-1}{i-1} = n \sum_{i=0}^{n-1} \binom{n-1}{i} = n \cdot 2^{n-1}$$

By induction, (1) $n = 1$ is trivial; (2) Given that $\sum_{i=0}^{n} i \binom{n}{i} = n \cdot 2^{n-1}$,

$$\begin{aligned}
\sum_{i=0}^{n+1} i \binom{n+1}{i} &= \sum_{i=0}^{n} i \binom{n}{i} + \sum_{i=1}^{n+1} i \binom{n}{i-1} \\
&= n \cdot 2^{n-1} + \sum_{i=0}^{n} (i+1) \binom{n}{i} \\
&= n \cdot 2^{n-1} + \sum_{i=0}^{n} i \binom{n}{i} + \sum_{i=0}^{n} \binom{n}{i} \\
&= 2n \cdot 2^{n-1} + 2^{n} \\
&= (n+1)\, 2^{n}
\end{aligned}$$

LPV 2.5.7

We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

Answer: If not, let a_i be the i-th number s.t. $a_i < a_{i+1}$. Then we have $a_i + 28 \leq a_{i+1}$, thus $1036 < a_1 + 1036 = a_1 + 26 \times (38 - 1) \leq a_{38} < 1000$, and that is impossible.

Special Problem 1

Consider the iPOD problem discussed in class with n songs in the storage. After $k = 2n$ songs were randomly selected and played, the probability of some song having been played (at least) twice is approximately $1 - e^{-1} \approx 0.63$. In this problem, we want to calculate the probability of hearing any song repeated (at least) three times. More precisely, let $p_{n,k}$ be the probability that, after k random songs were selected and played, some song has been played at least three times. What is the value of $p_{n,k}$ for $n = 10000$ and $k = 144$? (Your answer needs to be very close to the true value, even though you do not need to have a completely rigorous proof.)

Answer: I got $p_{n,k} = \frac{k!}{n^k} \sum_i \frac{1}{2^i} \binom{n}{i} \binom{n-i}{k-2i}$, but i have no way to compute it effectively.

Special Problem 2

Construct a probability space for the two envelopes problem discussed in class. Define a random variable X corresponding to the concept of gain if one decides to switch to the other envelope. Prove that the expected gain $E\,(X)$ is equal to 0.

Answer: Let $\{x, 2x\}$ be the cash in two envelopes. One's gain is $(2x - x)$ or $(x - 2x)$ (depends on which envelop is more valuable). So we get $E(X) = \sum_{x_0 \in [0,M)} P(x = x_0) \left(\frac{1}{2}(2x_0 - x_0) + \frac{1}{2}(x_0 - 2x_0)\right) = 0$

Special Problem 3

A 4 by 4 box is filled with integers $1, 2, \ldots, 15$ with the lower-right corner initially left empty. At any time there is exactly one cell empty. At each step you may move to the empty cell one of its adjacent number. Question: Prove that the second configuration cannot be reached from the first configuration.

Answer: Define $R(x_1, x_2, \ldots, x_n) = \#\{(x_i, x_j) | i < j, x_i > x_j\}$. Number the box from 1 to 16. Treat the empty box as being filled with 16. Let $x_i = j$ if box i is filled with number j. Each operation changes R from an odd number to even, or from an even to odd. Notice that R is even in the first checkboard, and odd in the second one. So transform from the first one to the second one needs odd numbers of operations. However, the empty box will have an odd number instead of 16 if we perform odd numbers of operations. So we have proved that the second configuration cannot be reached from the first configuration.

Acknowledgement: Answers here are all by myself.