Mathematics for Computer Science: Homework 10

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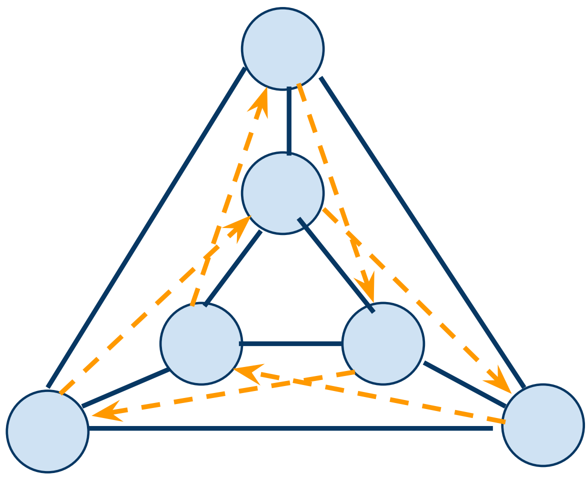
LPV 12.3.1

Is the complement of the cycle of length 6 (C_6) a planar graph?

Answer:



LPV 12.3.3

Does the ``picture frame'' polyhedron in Figure 12.6 in LPV satisfy Euler's Formula?

Answer:
No. Euler's Formula holds only for solids that can be blown up to a sphere. In this case, $V = 12$, $E = 24$, $F = 12$, $V + F - E = 0 \neq 2$.

LPV 12.3.5

Using Euler's Formula, show that the Petersen graph is not planar.

Answer: Suppose we can draw it in the plane. $V = 10$, $E = 15$, using Euler's Formula we get $F = 7$. In this graph, each cycle have at least 5 edges, and 2 faces share one edge. So we need at least $\frac{5\times 7}{2} = 17.5 > 15$ edges. That's impossible.

LPV 13.4.5

Let $G_n$be the graph arising from K_n by omitting the edges of a Hamiltonian cycle. Determine the chromatic number of G_n.

Answer: $\chi(G_n) = \left[\frac{n+1}{2}\right]$.

Assume we delete $V_1V_2\ldots V_nV_1$ from K_n. $V_k \to \left[\frac{k+1}{2}\right]$ is a legal coloring. A $\left\{V_{2k-1}|1 \le k \le \left[\frac{n+1}{2}\right]\right\}$ induced subgraph of G_n is a complete graph with $\left[\frac{n+1}{2}\right]$ points. We need at least $\left[\frac{n+1}{2}\right]$ colors for this subgraph.

LPV 13.4.7

If every face of a planar map has an even number of edges, then the graph is bipartite.

Answer:

Lemma.1 If every face of a planar map has an even number of edges, then each cycle in this graph contains even number of edges.

Proof If the cycle is a face, then it contains even number of edges. If not, there's some paths in it, each connecting two points. Use induction with the number of paths in cycle, then we can split it into two cycles by doubling a path in it, and each small cycle have fewer pathes in it. By inductive assumption, this two both contains even number of edges. Add them and remove two times the length of the path, we get that the original cycle contains even number of edges.

Lemma.2 If each cycle in a graph contains even number of edges, then the graph is bipartite.

Proof Pick a forest in the graph, and color each tree of this forest using two colors. Adding remaining edges sequencely to this forest cannot cause this coloring to fail since there's no odd cycles.

LPV 13.4.8

If every node of a planar map has even degree, then the faces can be 2-colored.

Answer:

This is the dual proposition of LPV 13.4.7.

Special Problem 1

The Sequential Coloring Algorithm may perform poorly on graphs that are not disk graphs. Prove that, for each n, there exists (G, σ), where graph G has n vertices and σ is an ordering of the vertices, such that the number used by the Sequential Algorithm is more than $\Omega(n^c)$ times $\chi(G)$ (where $c > 0$ is some fixed constant independent of n).

Answer:

Let $G_n = \{V = \{V_1, V_2, \ldots, V_n\}, E = \{(V_{2k_1-1}, V_{2k_2})\, |1 \le 2k_1 - 1 \le n, 1 \le 2k_2 \le n, k_1 \ne k_2\}\}$, $\sigma = (1, 2, \ldots, n)$. According to the Sequential Algorithm, $V_1 \to 1$, $V_2 \to 1$, $V_3 \to 2$ (Adj. with V_2), $V_4 \to 2$ (Adj. with V_1), and by induction, $V_{2k-1} \to k$, $V_{2k} \to k$. Thus we use total of $\left[\frac{n+1}{2}\right]$ colors. However, we can color $V_{2k-1} \to 1$,

$V_{2k} \to 2$ with 2 colors, thus $\chi(G) = 2$. $\frac{\left[\frac{n+1}{2}\right]}{2} = \Omega(n)$. (Here $c = 1$ & we can prove $c \leq 1$ because Sequential Algorithm cannot use more than n colors.)

Special Problem 2

Let $n \geq 5$ be any integer. Consider the undirected graph $G_n = (V_n, E_n)$, where $V_n = \{0, 1, 2, \ldots, n-1\}$, and E_n is the set of all pairs $\{i, j\}$ satisfying $i \neq j$ and $(i - j) \mod n \in \{1, 2, -1, -2\}$. Thus, the degree of each vertex is 4.

(a) What is the chromatic number $\chi(G_n)$?

(b) Is G_n a planar graph?

Answer:

(a) $\chi(G_n) = \begin{cases} 3 & n > 5, n \bmod 3 = 0 \\ 4 & n > 5, n \bmod 3 \neq 0 \\ 5 & n = 5 \end{cases}.$

Proof of $n = 5$. $G_5 = K_5 \Rightarrow \chi(G_5) = 5$.

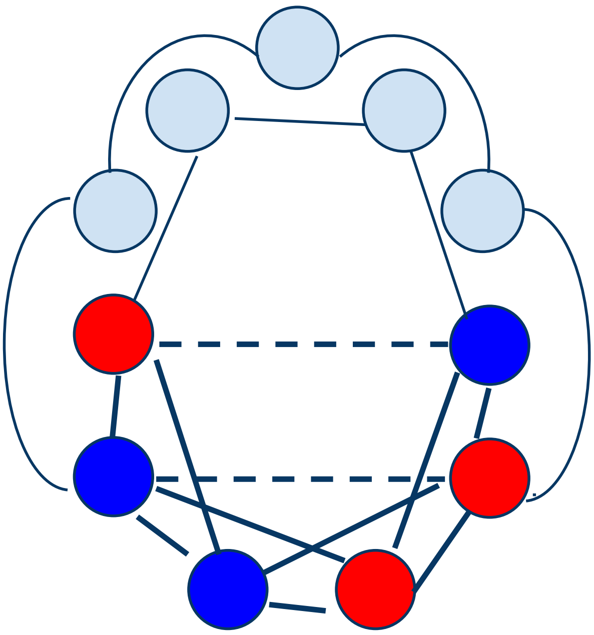
Proof of $n > 5, n \bmod 3 = 0$. Let $V_k \to k \bmod 3$ and total of 3 colors are used. We need at least 3, because $K_3 = \{V_0V_1, V_0V_2, V_1V_2\} \subset G_n$.

Proof of $n > 5, n \bmod 3 \neq 0$. For $n = 3m + 1$, let $V_k \to \begin{cases} k \bmod 4 & k < 4 \\ (k-1) \bmod 3 & k \geq 4 \end{cases}$ and total of 4 colors are used. For $n = 3m + 2$, let $V_k \to \begin{cases} k \bmod 4 & k < 8 \\ (k-2) \bmod 3 & k \geq 8 \end{cases}$ and total of 4 colors are used. On the other hand, $K_3 = \{V_0V_1, V_0V_2, V_1V_2\} \subset G_n$, so we need at least 3. Suppose we use exactly 3 colors. Let $V_0 \to C_0$, $V_1 \to C_1$, $V_2 \to C_2$. Then the color of V_3 must be C_0 (Adj. with V_1, V_2), $V_4 \to C_1$ (Adj. with V_2, V_3), by induction $V_k \to k \bmod 3$. Thus $\begin{cases} V_{n-1} \to (n-1) \bmod 3 = 0 & n \bmod 3 = 1 \\ V_{n-2} \to (n-2) \bmod 3 = 0 & n \bmod 3 = 2 \end{cases}$. But $V_0 \to 0$ and they're both connected to V_0. That's impossible. So we have to use at least 4 colors.

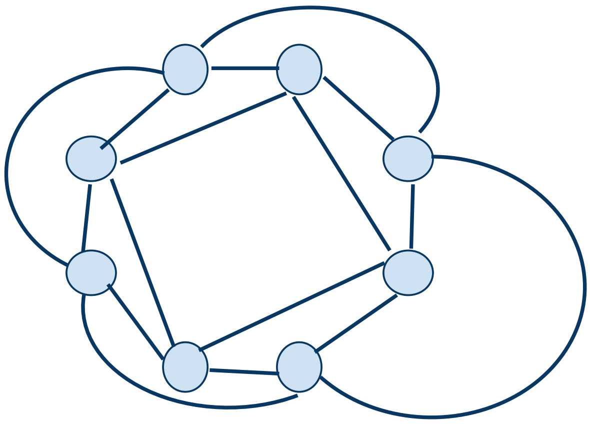
(b) Yes for even n, no for odd n.

Proof of $n = 5$. $G_5 = K_5$ is not planar.

Proof of $n \geq 7$, odd. Remove some edges from G_n, and we get a graph like this:



Red points and blue points is $K_{3,3}$. (The dashed lines mean indirect connections.)
Proof of $n \geq 6$, even. We can draw G_n in a plane like this:



Acknowledgement: Answers here are all original.