Mathematics for Computer Science: Homework 11

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LPV 11.3.2

Given $n \geq 3$ lines in the plane in general position, prove that among the regions they divide the plane into, there's at least one triangle.

Answer:

Use induction. When $n = 3$, these three lines did not go through a point, so there is a triangle. When we add a new line to the plane, if this line did not intersect with our old triangle, then the triangle remains. Or it divides the triangle into two parts, and the part with exactly one vertex of the old triangle is a new triangle.

LPV 11.3.6

Into how many parts can n circles divide the plane, maximum and minimum?

Answer:

Add circles one by one to a plane with a single circle. The k-th circle in this plane intersects with at most $k-1$ existing circles, each with two common points, and at least none, thus divides at most $2(k-1)$ regions into two pieces, and at least 1. So there will be at most $2 + \sum_k 2(k-1) = 2 + k(k+1) - 2k = k^2 - k + 2$ parts, and at least $2 + (k-1) = k+1$ parts.

LPV 11.3.7

Prove that 6 points in the plane, no 3 on a line, form at least 3 convex quadrilaterals.

Answer:

Lemma.1 If we are given five points in the plane such that no three of them are on a line, then we can always find four points among them that form the vertices of a convex quadrilateral.

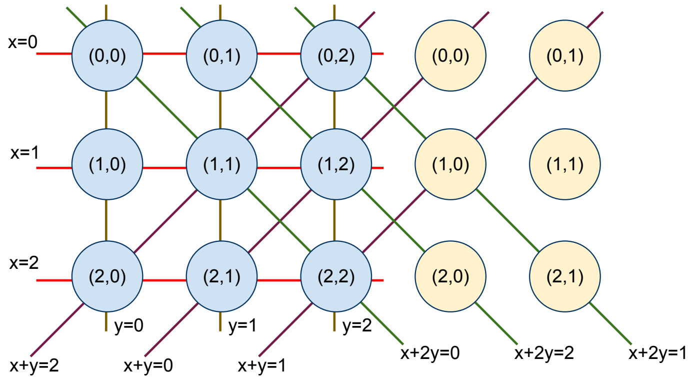
Proof See the LPV book.

Let the points be V_i, and in $A_i = \{V_k | 1 \leq k \leq 6\} \setminus \{V_i\}$ we get a convex quadrilateral Q_i. Let the number of quadrilaterals be x. Since each quadrilateral can only appear 2 times (e.g. $ABCD$ in $ABCDE$ & $ABCDF$) in $\{Q_i\}$, we have $6 = \#\{Q_i\} \leq 2x$, thus $x \geq 3$.

LPV 14.6.3

Verify that the Tictactoe plane is the same as the affine plane over the 3-element field.

Answer: Label the Tictactoe plane like this , and then we get a 3-element field.



LPV 14.6.5

In the game SMALLSET,
(a) What is the number of SETS?
(b) Show that for any two cards there is exactly one third card that forms with them a SET.
(c) What is the connection between this game and the affine space over the 3-element field?
(d) Prove that at the end of the game, either no cards or at least 6 cards remain.

Answer:
(a) Let it be a 3×3 cube, thus 13 directions, each with 9 ``lines'' (SETS). So 117.
(b) $ax_1 + bx_2 + cx_3 = d \pmod 3$ has only one solution $x_1 \equiv a^{-1}(d - cx_3 - bx_2)$, when x_2, x_3 is provided. Thus 3 coordinates is fully-determinated.
(c) This cube IS an affine space over the 3-element field. Each SET is in fact a line.
(d)

Lemma.1 Three points are in a SET iff $\sum x, \sum y, \sum z$ is zero mod 3.

Proof Notice $x + (x+1) + (x+2) \equiv 0$ or $x + x + x \equiv 0$, and the solution is unique for each x_1, x_2. A brute force proof also works.

Suppose there is exactly 3 cards left. Since the sum of all x, all y, all z is 0 mod 3. And all previous SETS have x, y, z sum to 0 mod 3. So the last 3 cards must have x, y, z sum to 0 mod 3. Then it is a SET.

LPV 14.6.7

Consider the prime field with 13 elements. For every two numbers x and y in the field, consider the triple $\{x + y, 2x + y, 3x + y\}$ of elements of the field. Show that these triples form a block design, and compute its parameters.

Answer:
There's 13 elements, and 78 sets.
Each element (let it be t) belongs to $\{t-2, t-1, t\}$, $\{t-1, t, t+1\}$, $\{t, t+1, t+2\}$, $\{t-4, t-2, t\}$, $\{t-2, t, t+2\}$, $\{t, t+2, t+4\}$, $\{t-6, t-3, t\}$, $\{t-3, t, t+3\}$, $\{t, t+3, t+6\}$, $\{t-8, t-4, t\}$, $\{t-4, t, t+4\}$,

$\{t, t+4, t+8\}$, $\{t-10, t-5, t\}$, $\{t-5, t, t+5\}$, $\{t, t+5, t+10\}$, $\{t-12, t-6, t\}$, $\{t-6, t, t+6\}$, $\{t, t+6, t+12\}$, totally 18 sets. 18 is a constant here.

Each set contains 3 elements. 3 is a constant here.

For x, y, they belongs to $\{2x-y, x, y\}$, $\{x, y, 2y-x\}$, $\{x, 7x+7y, y\}$, totally 3 sets. 3 is a constant here.

So this is a block design with $v = 13$, $b = 78$, $k = 3$, $r = 18$, $\lambda = 3$.

LPV 14.6.10

Consider the addition table of the ``Days of the Week'' number system in Section 6.8. Show that this table is a Latin square. Can you generalize this observation?

Answer: The square is like this, where Su = 0, Mo = 1, ...

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

There is exactly one $0, 1, 2, 3, 4, 5, 6$ in each row and each colomn.

In general, $a_{ij} = (i + j) \mod n$ for any n is a latin square.

LPV 14.6.11

Describe the code you get from the projective plane over the 3-element field, analogously to the Fano code. How much error correction/detection does it provide?

Answer:

Let each point correspond to a position in the codewords. (So the codewords will consists 13 bits) Each line will provide two codewords, one in which we put 1 on the line, and 0 off the line, and the other one otherwise. Add the all-zero and all-one string. We get 28 codewords.

Change a ``black'' line to a ``white'' line needs at least 5 errors, and a ``black'' to a ``black'' needs at least 6. Change all-zero or all-one to another valid codeword needs 4. Thus it is 3-error-detecting, and 1-error-correcting.

If we exclude the all-zero and all-one from our code, it will be 4-error-detecting, and 2-error-correcting. But now we have only 26 codewords.

Special Problem

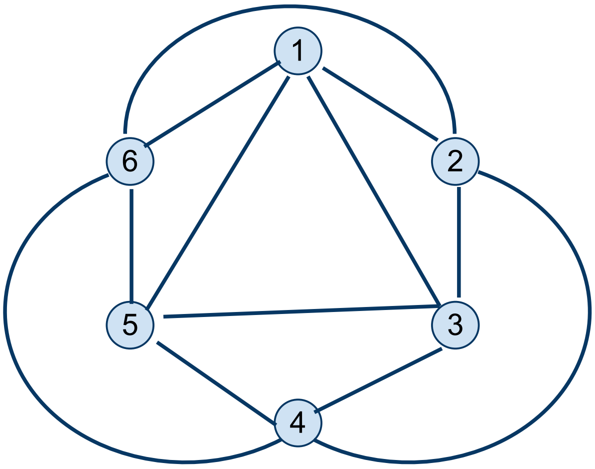
Let $D = (V, E)$ be the graph defined by $V = \{1, 2, 3, \ldots, 6\}$, and $E = \{\{i, j\} \,|1 \le i < j \le 6, j = 3 + i\}$. (That is, D is obtained from the complete graph K_6 by deleting from it a perfect matching. Thus $|E| = 12$.)

(a) Prove that D is a planar graph.

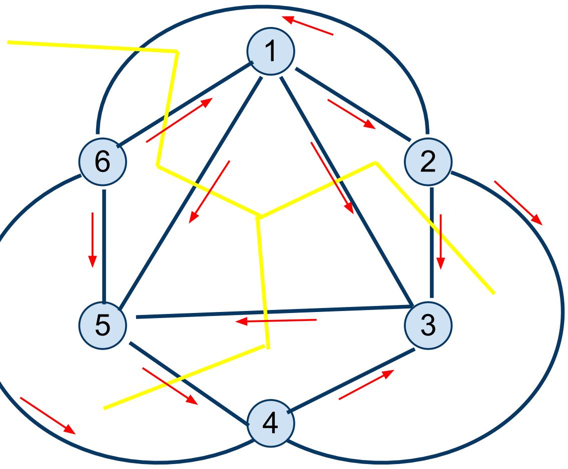
(b) Use Kasteleyn theorem to determine the number of perfect matchings in D.

Answer:

(a) Draw it like this.



(b) This is a Kasteleyn's orientation of D.



with the Kasteleyn's matrix,

$$K = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The number of perfect matching in D is $\sqrt{\det(K)}$.

Acknowledgement: Answers here are all original.