

# Mathematics for Computer Science: Homework 2

Instructed by *Andrew C. Yao & Wang Yuexuan*

Due on March 18, 2010

**Zhang Kunwei** J92 2009011269

## LPV 3.8.5

Find the value of  $k$  for which  $k \binom{99}{k}$  is largest.

**Answer:**  $k \binom{99}{k} = 99 \binom{98}{k-1}$ , which is maximum when  $k - 1 = 49$ , i.e.  $k = 50$ .

## LPV 3.8.8

Prove the following identities:

$$\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$$
$$\sum_{k=0}^n \binom{n}{k} \binom{k}{m} = \binom{n}{m} 2^{n-m}$$

**Answer:**

Compare the coefficients of  $x^m$  in  $(1+x)^n \frac{1}{1+x} = (1+x)^{n-1}$ , we get  $\sum_{k=0}^m \binom{n}{k} (-1)^{m-k} = \binom{n-1}{m}$ , i.e.  $\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$ .

For another one, we have:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \binom{k}{m} &= \sum_{k=0}^n \frac{n!k!}{k!(n-k)!m!(k-m)!} \\ &= \sum_{k=0}^n \frac{n!(n-m)!}{(n-m)!(n-k)!m!(k-m)!} \\ &= \sum_{k=0}^n \binom{n}{m} \binom{n-m}{n-k} \\ &= \binom{n}{m} \sum_{k=0}^n \binom{n-m}{k} \\ &= \binom{n}{m} 2^{n-m} \end{aligned}$$

**LPV 3.8.14**

Let  $n$  be a positive integer divisible by 3. Use Stirling's formula to find the approximate value of  $\binom{n}{n/3}$ .

**Answer:**

$$\begin{aligned} \binom{n}{n/3} &= \frac{n!}{\left(\frac{1}{3}n\right)! \left(\frac{2}{3}n\right)!} \\ &\sim \frac{\sqrt{2\pi n} (n/e)^n}{\sqrt{2\pi \frac{1}{3}n} \left(\frac{1}{3}n/e\right)^{\frac{1}{3}n} \sqrt{2\pi \frac{2}{3}n} \left(\frac{2}{3}n/e\right)^{\frac{2}{3}n}} \\ &= \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi} \left(\frac{1}{3}n\right)^{\frac{1}{3}n+\frac{1}{2}} \left(\frac{2}{3}n\right)^{\frac{2}{3}n+\frac{1}{2}}} \\ &= (2\pi)^{-\frac{1}{2}} n^{-\frac{1}{2}} 3^{n+1} 2^{-\frac{2}{3}n-\frac{1}{2}} \\ &= \frac{3}{2\sqrt{\pi}} \left(\frac{3}{\sqrt[3]{4}}\right)^n \end{aligned}$$

**LPV 4.3.7**

How many subsets does the set  $\{1, 2, \dots, n\}$  have that contain no three consecutive integers? Find a recurrence.

**Answer:** Let it be  $F_n$ . There are  $F_{n-3}$  such subsets containing  $n$  and  $n-1$ ,  $F_{n-2}$  containing  $n$  but not  $n-1$ ,  $F_{n-1}$  not containing  $n$ . So we get  $F_n = F_{n-3} + F_{n-2} + F_{n-1}$  with  $F_1 = 2$ ,  $F_2 = 4$ ,  $F_3 = 8$ .

**LPV 4.3.14**

Recalling the Lucas numbers  $L = \{1, 3, \dots, L_{n-1} + L_{n-2}\}$ , prove the following identities.

- (a)  $F_{2n} = F_n L_n$
- (b)  $2F_{k+n} = F_k L_n + F_n L_k$
- (c)  $2L_{k+n} = 5F_k F_n + L_k L_n$
- (d)  $L_{4k} = L_{2k}^2 - 2$
- (e)  $L_{4k+2} = L_{2k+1}^2 + 2$

**Answer:**

LEMMA 1:

$$L_n = 2F_{n-1} + F_n$$

PROOF: It holds when  $n = 1, 2$ . Suppose it holds when  $n < k$ . Then we have  $L_k = L_{k-1} + L_{k-2} = 2F_{k-2} + F_{k-1} + 2F_{k-3} + F_{k-2} = 2F_{k-1} + F_k$ .

LEMMA 2:

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$$

PROOF: See (4.5) in LPV.

To prove (b):

$$\begin{aligned}
& F_k L_n + F_n L_k \\
&= 2F_k F_{n-1} + F_k F_n + 2F_n F_{k-1} + F_n F_k \\
&= 2(F_k F_{n-1} + F_k F_n + F_n F_{k-1}) \\
&= 2(F_k F_{n+1} + F_n F_{k-1}) \\
&= 2F_{k+n}
\end{aligned}$$

To prove (c):

$$\begin{aligned}
& 2L_{k+n} \\
&= 2F_{k+n} + 4F_{k+n-1} \\
&= 2(F_k F_{n+1} + F_{k-1} F_n) + 4(F_k F_n + F_{n-1} F_{k-1}) \\
&= 2(F_k F_n + F_k F_{n-1} + F_{k-1} F_n) + 4(F_k F_n + F_{n-1} F_{k-1}) \\
&= 5F_k F_n + F_k F_n + 2F_k F_{n-1} + 2F_{k-1} F_n + 4F_{n-1} F_{k-1} \\
&= 5F_k F_n + (F_k + 2F_{k-1})(F_n + 2F_{n-1}) \\
&= 5F_k F_n + L_k L_n
\end{aligned}$$

To prove (a): Let  $k = n$  in (b).

LEMMA 3: Let  $q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$ . Then

$$L_n = q_1^n + q_2^n$$

PROOF:  $L_n = 2F_{n-1} + F_n = 2 \frac{1}{\sqrt{5}} (q_1^{-1} q_1^n - q_2^{-1} q_2^n) + \frac{1}{\sqrt{5}} (q_1^n - q_2^n) = \frac{2q_1^{-1}+1}{\sqrt{5}} q_1^n - \frac{2q_2^{-1}+1}{\sqrt{5}} q_2^n = q_1^n + q_2^n$ .

LEMMA 4:

$$L_n^2 = L_{2n} + 2(-1)^n$$

PROOF:  $L_n^2 = (q_1^n + q_2^n)^2 = q_1^{2n} + q_2^{2n} + 2(q_1 q_2)^n = L_{2n} + 2(-1)^n$ .

To prove (d): Let  $n = 2k$  in (LEMMA 4).

To prove (e): Let  $n = 2k + 1$  in (LEMMA 4).

## Special Problem 1

In a New Year's party with  $2n$  people,  $k$  random names are picked to receive gifts. Assume that these  $2n$  people are actually  $n$  husband-wife couples. Let  $p_{n,k}$  be the probability that at least for one couple, both husband and wife win gifts. (a) Give a mathematical formula for  $p_{n,k}$ . (b) Show that there exists a constant  $C$  such that  $|p_{n,2\sqrt{n}} - (1 - e^{-1})| < Cn^{-\frac{1}{2}}$  for all  $n$ .

**Answer:**

$$p_{n,k} = 1 - \frac{2^k \binom{n}{k}}{\binom{2n}{k}} = 1 - 2^k \frac{n!}{(n-k)!} \cdot \frac{(2n-k)!}{(2n)!}$$

Let  $m = \sqrt{n}$ ,  $a_n = \sqrt{n} (p_{n,2\sqrt{n}} - (1 - e^{-1}))$ . Notice that  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$  when  $n \rightarrow \infty$ .

$$\begin{aligned}
\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \sqrt{n} (p_{n,2\sqrt{n}} - (1 - e^{-1})) = \lim_{m \rightarrow \infty} m (p_{m^2,2m} - (1 - e^{-1})) \\
&= \lim_{m \rightarrow \infty} m \left( 1 - 2^{2m} \frac{(m^2)!}{(m^2 - 2m)!} \cdot \frac{(2m^2 - 2m)!}{(2m^2)!} - 1 + e^{-1} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - 2^{2m} \frac{\sqrt{2\pi m^2} \left(\frac{m^2}{e}\right)^{m^2}}{\sqrt{2\pi (m^2 - 2m)} \left(\frac{m^2 - 2m}{e}\right)^{m^2 - 2m}} \cdot \frac{\sqrt{2\pi (2m^2 - 2m)} \left(\frac{2m^2 - 2m}{e}\right)^{2m^2 - 2m}}{\sqrt{2\pi (2m^2)} \left(\frac{2m^2}{e}\right)^{2m^2}} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - 2^{2m} \frac{(m^2)^{m^2 + \frac{1}{2}} (2m^2 - 2m)^{2m^2 - 2m + \frac{1}{2}}}{(2m^2)^{2m^2 + \frac{1}{2}} (m^2 - 2m)^{m^2 - 2m + \frac{1}{2}}} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - \left(\frac{m}{m-1}\right)^{-m^2} \left(\frac{m-1}{m-2}\right)^{m^2 - 2m + \frac{1}{2}} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - \left(1 + \frac{1}{m-1}\right)^{-m^2} \left(1 + \frac{1}{m-2}\right)^{m^2 - 2m + \frac{1}{2}} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - e^{-\frac{m^2}{m-1}} e^{\frac{m^2 - 2m + \frac{1}{2}}{m-2}} \right) \\
&= \lim_{m \rightarrow \infty} m \left( e^{-1} - e^{-1 + \frac{1}{2(m-2)} - \frac{1}{m-1}} \right) \\
&= -e^{-1} \lim_{m \rightarrow \infty} m \left( e^{\frac{1}{2(m-2)} - \frac{1}{m-1}} - 1 \right) \\
&= -e^{-1} \lim_{m \rightarrow \infty} m \left( \frac{1}{2(m-2)} - \frac{1}{m-1} \right) \\
&= -e^{-1} \lim_{m \rightarrow \infty} \left( \frac{m}{2(m-2)} - \frac{m}{m-1} \right) \\
&= -e^{-1} \left( \frac{1}{2} - 1 \right) \\
&= \frac{1}{2} e^{-1}
\end{aligned}$$

Pick  $C_1$  s.t.  $C_1 > \frac{1}{2}e^{-1}$ . When  $\varepsilon = C_1 - \frac{1}{2}e^{-1}$ ,  $\exists N$  s.t. when  $n > N$ ,

$$|a_n - a_\infty| < \varepsilon \Rightarrow |a_n| - |a_\infty| < \varepsilon \Rightarrow |a_n| \leq C_1$$

Let  $C = \max(\max_{1 \leq k \leq N} |a_k|, C_1)$ , we have  $|a_n| \leq C$  for all  $n$ . Thus  $|\sqrt{n} (p_{n,2\sqrt{n}} - (1 - e^{-1}))| < C$ ,  $|p_{n,2\sqrt{n}} - (1 - e^{-1})| < Cn^{-\frac{1}{2}}$ .

## Special Problem 2

In class we discussed the Red-Blue Hat Problem for three participants, and a strategy was found by students that achieves a winning probability of 75%. Can you prove that this is optimal, i.e. no strategy can have a larger winning probability than 75%?

**Answer:** We prove the winning probability of the generalized  $n$ -people problem has an upper bound,  $\frac{n}{1+n}$ . Let  $R$  = Red,  $B$  = Blue,  $\bar{R} = B$ ,  $\bar{B} = R$ ,  $+1$  = Correct,  $0$  = Pass,  $-1$  = Wrong. We can then make a chart for each feasible strategy, e.g. the “75%” strategy for the 3-people problem are shown below.

Color	Guess	Result			
		$G_1$	$G_2$	$G_3$	Final
$X_1X_2X_3$	$Y_1Y_2Y_3$				
RRR	BBB	-1	-1	-1	Fail
RRB	00B	0	0	1	Succeed
RBR	0B0	0	1	0	Succeed
BRR	B00	1	0	0	Succeed
RBB	R00	1	0	0	Succeed
BRB	0R0	0	1	0	Succeed
BBR	00R	0	0	1	Succeed
BBB	RRR	-1	-1	-1	Fail

Let  $G_k(\vec{X})$  be the value of  $G_k$  in the row with color  $\vec{X}$ .

LEMMA 1:  $G_k(X_1X_2\dots X_k\dots X_n) + G_k(X_1X_2\dots \bar{X}_k\dots X_n) = 0$

PROOF: One cannot distinguish two cases here, so he will give the same reaction. If he gives a guess, one will be +1, the other will be -1. If he gives a "pass", both will be 0. But the sum is always 0.

LEMMA 2: The sum of all  $G$  is 0.

PROOF: In column  $k$ , pair each  $X_1X_2\dots X_k\dots X_n$  with  $X_1X_2\dots \bar{X}_k\dots X_n$ . Each pair adds 0 to the total sum of column  $k$ . So the sum of any column is 0, thus the sum of all  $G$  is 0.

Now let  $x$  be the number of cases with "Succeed" in final results. Each row in these  $x$  rows has a minimum sum of 1 (in which one is correct and  $n-1$  are wrong). Each failed row has a minimum sum of  $-n$  (in which everyone was wrong). So we have  $0 \geq x + (-n)(2^n - x)$ , thus  $p = \frac{x}{2^n} \leq \frac{n}{1+n}$ .

For  $n = 3$ , we have  $p \leq \frac{3}{4} = 75\%$ .

### Special Problem 3

A binary string  $\alpha$  is said to be balanced if the number of 1's in the first half of  $\alpha$  is exactly equal to the number of 1's in the second half. (For example, 00100110 is not balanced since there are two 1's in the second half and only one 1 in the first half; on the other hand, the string 01001000 is balanced.) Prove that there are exactly  $\binom{2n}{n}$  balanced strings of length  $2n$ .

**Answer:** Select  $n$  places out of  $2n$  and mark with 1, others are marked with 0. The  $k$ -th 1 has  $2n+1-k$  choices, and each of the balanced strings is calculated with  $n!$  times. So it turns out to be  $\binom{2n}{n}$  balanced strings.

### Special Problem 4

Consider a sequence of  $2n$  people in a line at a cashier. Suppose  $n$  of the people pay 1 yuan each and  $n$  of the people get 1 yuan each. A paying pattern is a binary sequence  $\sigma = a_1a_2\dots a_{2n}$  with exactly  $n$  1's and  $n$  0's; the interpretation is that  $a_j = 1$  if person  $j$  pays 1 yuan, and  $a_j = 0$  otherwise. Note that there are exactly  $\binom{2n}{n}$  paying patterns. Prove that the number of paying patterns in which the cashier never goes in debt (i.e., at every stage at least as many people have paid in 1 yuan as were paid out 1 yuan) is equal to  $\binom{2n}{n} - \binom{2n}{n+1}$ . (Hint: Show a one-to-one correspondence between paying patterns where at some stage the cashier goes at least 1 yuan in debt and all binary sequences of length  $2n$  with exactly  $n+1$  1's.)

**Answer:**

Let  $f(x) = \begin{cases} -1 & x = 0 \\ 1 & x = 1 \end{cases}$ . Consider a correspondence  $F(\sigma) = \sigma_1$  between paying patterns where at some stage the cashier goes at least 1 yuan in debt and all binary sequences of length  $2n$  with exactly  $n+1$  0's. Given a paying pattern  $\sigma = a_1a_2\dots a_{2n}$ , in which  $k$  is the minimal of all  $m$  s.t.  $\sum_{i=1}^m f(a_i) = -1$ , i.e. the cashier goes 1 yuan in debt for the first time after stage  $k$ . Map  $\sigma$  to  $\sigma_1 = a_1a_2\dots a_k\bar{a}_{k+1}\dots a_{2n}$ . There are exactly  $\frac{k+1}{2} + \frac{2n-k+1}{2} = n+1$  0's in  $\sigma_1$ . On the other hand, given a sequence  $\sigma_1 = b_1b_2\dots b_kb_{k+1}\dots b_{2n}$  with  $n+1$

in it, let  $k$  be the minimalist of all  $m$  s.t.  $\sum_{i=1}^m f(b_i) = -1$ , and then we get  $F^{-1}(\sigma_1) = b_1 b_2 \dots b_k \overline{b_{k+1} \dots b_{2n}}$ . So  $F$  is an one-to-one correspondence. Thus the number of paying patterns in which the cashier never goes in debt is equal to  $\binom{2n}{n} - \binom{2n}{n+1}$ .

## Special Problem 5

Let  $c$  be a real number with standard binary expansion  $c = 0.x_1 x_2 \dots x_n$ ,  $x_n = 1$ . Let  $g(c) = \sum_{0 \leq k \leq n} k 2^{-k} + n 2^{-n}$ .

- Prove that there is a  $c$ -coin toss tree  $A$  with cost  $g(c)$ .
- Prove that every  $c$ -coin toss tree must have cost greater than or equal to  $g(c)$ .

### Answer:

According to the notation in this problem's statement, we use  $|w|$  for the length of string  $w$ .

(a) Let  $A = \{ "0" \rightarrow x_1, "10" \rightarrow x_2, "110" \rightarrow x_3, \dots, "11 \dots 10" \rightarrow x_{n-1}, "11 \dots 110" \rightarrow x_n, "11 \dots 111" \rightarrow 0 \}$ , then the cost of  $A$  is  $\sum_{0 \leq k \leq n} k 2^{-k} + n 2^{-n} = g(c)$ .

(b) Let  $C(X)$  be the cost of tree  $X$ , and we want to prove  $C(x) \geq g(c)$ . We treat all nodes which have a prefix (not itself) in  $W$  as not being in the tree, and do induction over  $n$ . It's trivial when  $n = 1$ . If it holds when  $n < p$ , now we can prove it holds when  $n = p$ :

If the depth of the tree  $X$  is at most  $p - 1$ , then  $c = \sum_{f(w)=1} 2^{-|w|} = k_1 2^{n-1}$  i.e.  $x_n = 0$ . That is impossible. So the depth of  $X$  is at least  $p$ , i.e. there exists an at-least- $p - 1$ -depth node with 2 childs who are leaves. This tree without these two leaves gives a  $(c - 2^{-p})$ -coin toss tree, whose cost is  $C(X) - p 2^{-p} - p 2^{-p} + (p - 1) 2^{-(p-1)}$ , which is greater than or equal to  $g(c - 2^{-p}) = \sum_{0 \leq k \leq p} k 2^{-k} + (p - 1) 2^{-(p-1)}$  according to the inductive assumption. Thus  $C(X) \geq \sum_{0 \leq k \leq p} k 2^{-k} + (p - 1) 2^{-(p-1)} + p 2^{-p} + p 2^{-p} - (p - 1) 2^{-(p-1)} = \sum_{0 \leq k \leq n} k 2^{-k} + n 2^{-n} = g(c)$ .

**Acknowledgement:** Answers here are all by myself.