# Mathematics for Computer Science: Homework 4

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Due on April 1, 2010

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## LPV 6.10.22

We are given $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that there are two numbers among them such that one divides the other.

**Answer:** Let set $X_i = \left\{m | m = 2^k (2i-1)\right\}$, $1 \le i \le n$. There are two numbers in the same set, thus one divides the other.

## LPV 6.10.23

What is the number of positive integers not larger than 210 and not divisible by 2, 3, or 7?

**Answer:** $2|x$, $3|x$, $7|x \Leftrightarrow 42|x$. There are $\left[\frac{210}{42}\right] = 5$ such numbers.

## Special Problem 1

Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr\{X_i = 1\} = p_i$. Let $X = \sum_{1 \le i \le n} X_i$ and $\mu = E(X)$. In class we derived one version of the Chernoff Bounds regarding the probability that $X > (1+\delta)\mu$. Here you are asked to prove the following bounds in a similar way:

(a) For $0 < \delta < 1$, $\Pr\{X \le (1-\delta)\mu\} \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}$

(b) Assume that $p_i = \frac{1}{2}$ for all $i$. Prove the stronger bound that $\Pr\left\{\left|X - \frac{n}{2}\right| > a\right\} \le 2e^{\frac{-2a^2}{n}}$.

**Answer:**

(a)

$$\begin{aligned}
E\left(e^{kX}\right) &= E\left(e^{kX_1} e^{kX_2} \ldots e^{kX_n}\right) \\
&= E\left(e^{kX_1}\right) E\left(e^{kX_2}\right) \ldots E\left(e^{kX_n}\right) \\
&= \prod_i \left(p_i \left(e^k - 1\right) + 1\right) \\
&\le \left(\frac{\sum_i \left(p_i \left(e^k - 1\right) + 1\right)}{n}\right)^n \\
&= \left(1 + \left(e^k - 1\right) \frac{\mu}{n}\right)^n
\end{aligned}$$

$$\begin{aligned} &= \left(1+\left(e^k-1\right)\frac{\mu}{n}\right)^{\frac{n}{\left(e^k-1\right)\mu}\cdot\left(e^k-1\right)\mu} \\ &\le e^{\left(e^k-1\right)\mu} \end{aligned}$$

Assume $k<0$. We have $\Pr\{X\le(1-\delta)\mu\}=\Pr\left\{e^{kX}\ge e^{k(1-\delta)\mu}\right\}\le\frac{E\left(e^{kX}\right)}{e^{k(1-\delta)\mu}}\le\frac{e^{\left(e^k-1\right)\mu}}{e^{k(1-\delta)\mu}}$. Let $k=\ln(1-\delta)$, we have $\Pr\{X\le(1-\delta)\mu\}\le\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}$.

(b)

Lemma 1: $e^t+1\le 2e^{t/2+t^2/8}$ for all $t>0$.

Proof: Let $k=e^t>1$, $f(t)=\frac{t}{2}+\frac{t^2}{8}+\ln 2-\ln(k+1)$, $f(0)=0$, $f'(t)=\frac{1}{2}+\frac{t}{4}-\frac{k}{k+1}$, $f'(0)=0$,

$$\begin{aligned} f''(t) &= \frac{1}{4}-\frac{k(k+1)+k^2}{(k+1)^2} \\ &= \frac{(k+1)^2-4k(k+1)+4k^2}{4(k+1)^2} \\ &= \frac{(k-1)^2}{4(k+1)^2} \\ &\ge 0 \end{aligned}$$

So we have $f(t)\ge 0$ for all $t>0$, thus Lemma 1.

Assume $k>0$. Here $\delta=\frac{2a}{n}$.

$$\begin{aligned} \Pr\left\{\left|X-\frac{n}{2}\right|>a\right\} &= 2\Pr\{X>(1+\delta)\mu\} \\ &= 2\Pr\left\{e^{kX}\ge e^{(1+\delta)\mu}\right\} \\ &\le 2\frac{E\left(e^{kX}\right)}{e^{k(1+\delta)\mu}} \\ &\le 2\frac{\left(1+\frac{\mu}{n}\left(e^k-1\right)\right)^n}{e^{k(1+\delta)\mu}} \\ &= 2\frac{\left(\frac{1+e^k}{2}\right)^n}{e^{k\left(\frac{1}{2}n+a\right)}} \\ &\le 2\frac{e^{\left(\frac{k}{2}+\frac{k^2}{8}\right)n}}{e^{k\left(\frac{1}{2}n+a\right)}} \\ &= 2e^{\frac{n}{8}k^2-ak} \end{aligned}$$

Let $k=\frac{4a}{n}$, $\Pr\left\{\left|X-\frac{n}{2}\right|>a\right\}\le 2e^{\frac{n}{8}k^2-ak}=2e^{-\frac{2a^2}{n}}$.

## Special Problem 2

Let $P_{2n}$ be the set of all $(2n)!$ permutations of $\{1,2,3,\ldots,2n\}$. For any $\sigma=(a_1,a_2,\ldots,a_{2n})\in P_{2n}$ , a pair of positions $(i,j)$ (where $i<j$) is called an inversion in $\sigma$ if $a_i>a_j$. Let $f(\sigma)$ be the permutation ob- tained from $\sigma$ by sorting the sublist of odd positions. That is, $f(\sigma)=(b_1,b_2,\ldots,b_2n)\in P_{2n}$, where $b_k=a_k$ for $k=2,4,6,\cdots,2n$ and $b_1<b_3<b_5<\ldots<b_{2n-1}$ is the sorted list of $a_1,a_3,\ldots,a_{2n-1}$. For a random $\sigma$ uniformly chosen from $P_{2n}$ , let $I_n$ be the random variable corresponding to the number of inversions in the permutation $f(\sigma)$. Do the following problems:

(a) Determine $\mathrm{E}(I_n)$.

(b) Determine $\mathrm{Var}(I_n)$.

**Answer:**

(a)

$$
\begin{aligned}
E(I_n) &= \frac{1}{2}\frac{n\,(n-1)}{2} + \frac{1}{n+1}\sum_{1\leq i\leq n}\left(\frac{(1+i)\,i}{2} + i\,(n-i)\right) \\
&= \frac{n\,(n-1)}{4} + \frac{1}{n+1}\left(\frac{1}{2}+n\right)\sum_{1\leq i\leq n} i - \frac{1}{2}\frac{1}{n+1}\sum_{1\leq i\leq n} i^2 \\
&= \frac{n\,(n-1)}{4} + \frac{1}{n+1}\left(\frac{1}{2}+n\right)\frac{n\,(1+n)}{2} - \frac{1}{2}\frac{1}{n+1}\frac{n\,(1+n)\,(1+2n)}{6} \\
&= \frac{n\,(n-1)}{4} + \left(\frac{1}{2}+n\right)\frac{n}{2} - \frac{n\,(1+2n)}{12} \\
&= \frac{7n^2-n}{12}
\end{aligned}
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(b) Bu Hui Zuo. -_-!!

**Acknowledgement:** Answers here are all original.