Mathematics for Computer Science: Homework 5

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Concrete Mathematics 8.18

A random variable X is said to have the Poisson distribution with mean μ if $\Pr(X = k) = e^{-\mu}\mu^k/k!$ for all $k \geq 0$.

(a) What is the pgf of such a random variable?

(b) What are its mean, variance, and other cumulants?

Answer:

(a) $G_X(z) = \sum_{k\geq 0} \frac{e^{-\mu}\mu^k}{k!} z^k = e^{\mu z - \mu}$;

(b) $\sum_{i\geq 1} \frac{\kappa_i}{i!} z^i = \ln G_X(e^z) = \ln e^{\mu e^Z - \mu} = \mu e^z - \mu = \sum_{i\geq 1} \frac{\mu}{i!} z^i$. Thus $\kappa_i = \mu$ for all $i > 0$.

Concrete Mathematics 8.20

Prove the general formulas for mean and variance of the time needed to wait for a given pattern of heads and tails.

Answer:

For the same reason of 8.67,

$$\begin{aligned} 1 + Nz &= N + G(z) \\ N &= \frac{G(z) - 1}{z - 1} \end{aligned}$$

For the same reason of 8.68,

$$\begin{aligned} N\frac{1}{\tilde{A}} &= G(z) \sum_{i=1}^{m} \left(\tilde{A}^{(m-i)}\right)^{-1} z^{m-i} \left[A^{(i)} = A_{(i)}\right] \\ N &= G(z) \sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right] \\ \frac{G(z) - 1}{z - 1} &= G(z) \sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right] \\ 1 &= G(z) \left(1 - (z-1) \sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right]\right) \\ G(z) &= \frac{z^m}{z^m - (z-1)\sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right]} \end{aligned}$$

Let

$$\begin{aligned} F(z) &= z^m - (1-z)\sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right] \\ &= z^m + \sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i+1} \left[A^{(i)} = A_{(i)}\right] - \sum_{i=1}^{m} \tilde{A}_{(i)} z^{m-i} \left[A^{(i)} = A_{(i)}\right] \end{aligned}$$

Then

$$\begin{aligned} F'(z) &= mz^{m-1} - \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i+1) z^{m-i} \left[A^{(i)} = A_{(i)}\right] + \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i) z^{m-i-1} \left[A^{(i)} = A_{(i)}\right] \\ F'(1) &= m - \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i+1) \left[A^{(i)} = A_{(i)}\right] + \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i) \left[A^{(i)} = A_{(i)}\right] \\ &= m - \sum_{i=1}^{m} \tilde{A}_{(i)} \left[A^{(i)} = A_{(i)}\right] \end{aligned}$$

$$\begin{aligned} F''(z) &= m(m-1) z^{m-2} - \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i+1)(m-i) z^{m-i-1} \left[A^{(i)} = A_{(i)}\right] \\ &\quad + \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i)(m-i-1) z^{m-i-2} \left[A^{(i)} = A_{(i)}\right] \\ F''(1) &= m(m-1) - \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i+1)(m-i) \left[A^{(i)} = A_{(i)}\right] \\ &\quad + \sum_{i=1}^{m} \tilde{A}_{(i)} (m-i)(m-i-1) \left[A^{(i)} = A_{(i)}\right] \\ &= m(m-1) - 2\sum_{i=1}^{m} \tilde{A}_{(i)} (m-i) \left[A^{(i)} = A_{(i)}\right] \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{Mean}(G) &= \operatorname{Mean}(z^m) - \operatorname{Mean}(F) \\ &= \sum_{i=1}^{m} \tilde{A}_{(i)} \left[A^{(i)} = A_{(i)}\right] \end{aligned}$$

$$\begin{aligned} \operatorname{Var}(G) &= \operatorname{Var}(z^m) - \operatorname{Var}(F) \\ &= \left(\sum_{i=1}^{m} \tilde{A}_{(i)} \left[A^{(i)} = A_{(i)}\right]\right)^2 - \sum_{i=1}^{m} \tilde{A}_{(i)} (2i-1) \left[A^{(i)} = A_{(i)}\right] \end{aligned}$$

Concrete Mathematics 8.26

Find the mean and variance of the number of l-cycles in a random permutation of n elements.

Answer:

Let $F_{n,l}$ be the pgf of the number of l-cycles. For the k-th coefficient of $F_{n,l,k}$, i.e. the probability of there are exactly k l-cycles in a random permutation of n elements, one can calculate it in such a way. First, make sure that there is a l-cycle, with probability $\frac{1}{n!}\binom{n}{l}(n-l)!\,(l-1)! = \frac{1}{l}$. Then, make sure that there are exactly $k-1$ l-cycles in a random permutation of $n-l$ elements, with probability $F_{n-l,l,k-1}$. Each permutation appears k times. (for k different selection of the first cycle) So we have $kF_{n,l,k} = \frac{1}{l}F_{n-l,l,k-1}$.

It follows up that

$$\begin{aligned} F'_{n,l}(z) &= \sum_k kF_{n,l,k}z^{k-1} \\ &= \sum_k \frac{1}{l}F_{n-l,l,k-1}z^{k-1} \\ &= \frac{1}{l}\sum_k F_{n-l,l,k}z^k \\ &= \frac{1}{l}F_{n-l,l}(z) \end{aligned}$$

Hence

$$\text{Mean}(F_{n,l}) = F'_{n,l}(1) = \frac{1}{l}F_{n-l,l}(z) = \begin{cases} \frac{1}{l} & n \ge l \\ 0 & n < l \end{cases}$$

$$\begin{aligned} \text{Var}(F_{n,l}) &= F''_{n,l}(1) + F'_{n,l}(1) - \left(F'_{n,l}(1)\right)^2 \\ &= \begin{cases} 0 & n < l \\ \frac{1}{l} - \frac{1}{l^2} & l \le n < 2l \\ \frac{1}{l} & 2l \le n \end{cases} \end{aligned}$$

Concrete Mathematics 8.29

Alice, Bill, and Computer flip a fair coin until one of the respective patterns $A =$ HHTH, $B =$ HTHH, or $C =$ THHH appears for the first time. What are each player's chances of winning?

Answer:

In this case, we have

$$\begin{aligned} N &= S_A(A:A) + S_B(B:A) + S_C(C:A) \\ N &= S_A(A:B) + S_B(B:B) + S_C(C:B) \\ N &= S_A(A:C) + S_B(B:C) + S_C(C:C) \end{aligned}$$

Plug everything into it, we get $S_A : S_B : S_C = 16 : 17 : 19$. Notice $S_A + S_B + S_C = 1$,

$$\begin{cases} S_A = \frac{16}{52} \\ S_B = \frac{17}{52} \\ S_C = \frac{19}{52} \end{cases}$$

Concrete Mathematics 8.46

Stefan Banach used to carry two boxes of matches, each containing n matches initially. Whenever he needed a light he chose a box at random, each with probability $\frac{1}{2}$, independent of his previous choices. After taking out a match he'd put the box back in its pocket (even if the box became empty - all famous mathematicians used to do this). When his chosen box was empty he'd throw it away and reach for the other box.

(a) Once he found that the other box was empty too. What's the probability that this occurs? (For $n = 1$ it happens half the time and for $n = 2$ it happens 3/8 of the time.) To answer this part, find a closed

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form for the generating function $P(w,z) = \sum_{m,n} p_{m,n} w^m z^n$, where $p_{m,n}$ is the probability that, starting with m matches in one box and n in the other, both boxes are empty when an empty box is first chosen. Then. find a closed form for $p_{n,n}$.

(b) Generalizing your answer to part (a), find a closed form for the probability that exactly k matches are in the other box when an empty one is first thrown away.

(c) Find a closed form for the average number of matches in that other box.

Answer:

(b)

$$p_{m,n,k} = \begin{cases} \frac{1}{2}p_{m-1,n,k} + \frac{1}{2}p_{m,n-1,k} & m \geq 0, n \geq 0 \\ 1 & (m=-1, n=k) \text{ or } (n=-1, m=k) \\ 0 & (m=-1, n\neq k) \text{ or } (n=-1, m\neq k) \end{cases}$$

leads to

$$P_k(w,z) = \frac{1}{2}wP_k(w,z) + \frac{1}{2}zP_k(w,z) + w^{-1}z^k + z^{-1}w^k - \frac{1}{2}w^{-1}z^{k+1} - \frac{1}{2}z^{-1}w^{k+1}$$

Thus

$$\begin{aligned}
P_k(w,z) &= \frac{w^{-1}z^k + z^{-1}w^k - \frac{1}{2}w^{-1}z^{k+1} - \frac{1}{2}z^{-1}w^{k+1}}{1 - \left(\frac{1}{2}w + \frac{1}{2}z\right)} \\
&= \left(w^{-1}z^k + z^{-1}w^k - \frac{1}{2}w^{-1}z^{k+1} - \frac{1}{2}z^{-1}w^{k+1}\right)\sum_i \left(\frac{1}{2}w + \frac{1}{2}z\right)^i \\
&= \left(w^{-1}z^k + z^{-1}w^k - \frac{1}{2}w^{-1}z^{k+1} - \frac{1}{2}z^{-1}w^{k+1}\right)\sum_i\sum_u w^u z^{i-u} 2^{-i}\binom{i}{u} \\
&= \left(w^{-1}z^k + z^{-1}w^k - \frac{1}{2}w^{-1}z^{k+1} - \frac{1}{2}z^{-1}w^{k+1}\right)\sum_{u,v} w^u z^v 2^{-u-v}\binom{u+v}{u} \\
&= \sum_{u,v} w^{u-1}z^{v+k}2^{-u-v}\binom{u+v}{u} + \sum_{u,v} w^{u+k}z^{v-1}2^{-u-v}\binom{u+v}{v} \\
&\quad -\frac{1}{2}\sum_{u,v} w^{u-1}z^{v+k+1}2^{-u-v}\binom{u+v}{u} - \frac{1}{2}\sum_{u,v} w^{u+k+1}z^{v-1}2^{-u-v}\binom{u+v}{v} \\
&= \sum_{u,v} w^u z^v 2^{-u-1-v+k}\binom{u+1+v-k}{u+1} + \sum_{u,v} w^u z^v 2^{-u+k-v-1}\binom{u-k+v+1}{v+1} \\
&\quad -\frac{1}{2}\sum_{u,v} w^u z^v 2^{-u-v+k}\binom{u+v-k}{u+1} - \frac{1}{2}\sum_{u,v} w^u z^v 2^{-u+k-v}\binom{u-k+v}{v+1} \\
&= \sum_{u,v} w^u z^v \cdot 2^{-u-v+k-1}\left(\binom{u+v-k+1}{u+1} + \binom{u+v-k+1}{v+1} - \binom{u+v-k}{u+1} - \binom{u-k+v}{v+1}\right) \\
&= \sum_{u,v} w^u z^v \cdot 2^{-u-v+k-1}\left(\binom{u+v-k}{u} + \binom{u+v-k}{v}\right)
\end{aligned}$$

$$p_{u,v,k} = 2^{-u-v+k-1}\left(\binom{u+v-k}{u} + \binom{u+v-k}{v}\right)$$

(a) Just plug $k=0$, $u=v=n$ in (b),

$$p_{n,n} = 2^{-2n}\binom{2n}{n}$$

(c)

$$\begin{aligned}
E_{n,n}(k) &= \sum_k k p_{n,n,k} \\
&= \sum_k k 2^{-n-n+k-1}\left(\binom{n+n-k}{n}+\binom{n+n-k}{n}\right) \\
&= \sum_k k 2^{k-2n}\binom{2n-k}{n} \\
&= \sum_k (n-k)\, 2^{-n-k}\binom{n+k}{n} \\
&= (2n+1)\sum_k 2^{-n-k}\binom{n+k}{n} - (n+1)\sum_k 2^{-n-k}\binom{n+1+k}{n+1} \\
&= \frac{2n+1}{2^{2n}}\binom{2n}{n} - 1
\end{aligned}$$

Concrete Mathematics 8.47

Find a closed form for the average number of diphages present, if we begin with a single diphage and irradiate the culture n times with single psi-particles.

Answer:

Let a_n be the average number of diphages present. Then $a_0 = 1$,

$$\begin{aligned}
a_n &= a_{n-1} - \frac{2a_{n-1}}{n+1} + 2\frac{n+1-2a_{n-1}}{n+1} \\
&= \frac{n-5}{n+1}a_{n-1} + 2
\end{aligned}$$

By some guesswork & induction, we get $a_n = \frac{2n+4}{7}$.

Concrete Mathematics 8.59

Are there patterns A and B of heads and tails such that A is longer than B, yet A appears before B more than half the time when a fair coin is being flipped?

Answer: Let $A = \text{HHTT}$, $B = \text{TTT}$. Then $\begin{cases} (A:A) = 8 \\ (A:B) = 3 \\ (B:A) = 0 \\ (B:B) = 7 \end{cases} \Rightarrow \frac{S_A}{S_B} = \frac{7}{5} > 1.$

Acknowledgement: Answers here are all original, except for the summation method used in (8.46), which is from Lemma (5.20) in Concrete Mathematics.