Mathematics for Computer Science: Homework 6

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Special Problem 1

Let σ be a permutation so that for each node $j \in \{0,1\}^n$ in the hypercube network, a packet v_j is to be routed to node $\sigma(j)$. For each node j, let $\rho_j = e_1 e_2 \ldots e_{l_j}$ be the path followed by packet v_j under the bit-fixing strategy. Now let i be any fixed node. Let S be the set of $j = i$ such that the paths ρ_j and ρ_i share at least one common edge.

(a) If $|S| \leq 4$, give a rigorous proof that the number of steps used in delivering packet v_i is no more than $i + |S|$.

(b) Prove that, for any $|S|$, the number of steps used in delivering packet v_i is no more than $i + |S|$.

Answer:

(a) & (b) Answer is in the other document.

Special Problem 2

Solve each of the following recurrence relations:

(a) $a_0 = 1$; $a_1 = 2$; $a_n = 4a_{n-1} - 3a_{n-2} + 3n + 1$ for all $n \geq 2$.

(b) $a_0 = 1$;$a_n = \frac{a_{n-1}}{1+3a_{n-1}}$ for $n \geq 1$.

Answer:

(a) Let $G(x) = \sum_i a_i x^i$, $G(x) = 4xG(x) - 3x^2 G(x) + \frac{3x}{(1-x)^2} + \frac{1}{1-x} + k_1 + k_2 x$,

$$\begin{aligned} G(x) &= \frac{3x + 1 - x + (1-x)^2 (k_1 + k_2 x)}{(1 - 4x + 3x^2)(1-x)^2} \\ &= \frac{t_0 + t_1 x + t_2 x^2 + t_3 x^3}{(1-3x)(1-x)^3} \\ &= \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3} \end{aligned}$$

Thus,

$$\begin{aligned} a_n &= A \cdot 3^n + B + C(n+1) + D(n+1)(n+2) \\ &= a \cdot 3^n + b + cn + dn^2 \end{aligned}$$

And we know $\{a_n\} = \{1, 2, 12, 52, \ldots\}$, we get

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 \\ 9 & 1 & 2 & 4 \\ 27 & 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \\ 12 \\ 52 \end{pmatrix} \\ (a, b, c, d) &= \frac{1}{8}(21, -13, -28, -6) \end{aligned}$$

Thus, $a_n = \frac{1}{8}\left(21 \cdot 3^n - 13 - 28n - 6n^2\right)$.
(b) Let $b_n = a_n^{-1}$, $b_0 = 1$,

$$\begin{aligned} b_n &= a_n^{-1} \\ &= \left(\frac{a_{n-1}}{1 + 3a_{n-1}}\right)^{-1} \\ &= \frac{1 + 3a_{n-1}}{a_{n-1}} \\ &= a_{n-1}^{-1} + 3 \\ &= b_{n-1} + 3 \end{aligned}$$

So we get $b_n = 3n + 1$, $a_n = \frac{1}{3n+1}$.

Special Problem 3

Let b_n be the number of different ways to tile completely a $2 \times n$ board by using only dominoes. It is clear that $b_1 = 1$; $b_2 = 2$; $b_n = b_{n-1} + b_{n-2}$ for $n \geq 3$. Thus the b_n's are just the Fibonacci numbers. Now let c_n be the number of different ways to tile a $3 \times n$ board by dominoes. Derive an explicit closed-form expression for c_n. (Note that clearly $c_n = 0$ if n is an odd integer.)

Answer:

Let d_n be the number of different ways to tile a $3 \times n + 1$ board by dominoes, e_n be the number of different ways to tile a $3 \times n + 2$ board by dominoes.

$$\begin{cases} c_n = c_{n-2} + 2e_{n-2} \\ d_n = e_{n-1} \\ e_n = c_n + d_{n-1} \end{cases} \Rightarrow \begin{cases} c_n = c_{n-2} + 2e_{n-2} \\ e_n = c_n + e_{n-2} \end{cases} \Rightarrow c_n = 4c_{n-2} - c_{n-4}.$$

Let $G(x) = \sum_i c_{2i} x^i$, $G(x) = 4xG(x) - x^2 G(x) + k_1 + k_2 x$

$$\begin{aligned} G(x) &= \frac{k_1 + k_2 x}{x^2 - 4x + 1} \\ &= \frac{A}{1 - \left(2 + \sqrt{3}\right)x} + \frac{B}{1 - \left(2 - \sqrt{3}\right)x} \end{aligned}$$

Thus $c_{2n} = A\left(2 + \sqrt{3}\right)^n + B\left(2 - \sqrt{3}\right)^n$. And we know $\{c_{2n}\} = \{1, 3, \ldots\}$,

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 2 + \sqrt{3} & 2 - \sqrt{3} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ (A, B) &= \frac{1}{6}\left(3 + \sqrt{3}, 3 - \sqrt{3}\right) \end{aligned}$$

Then $c_{2n} = \frac{1}{6}\left(\left(3 + \sqrt{3}\right)\left(2 + \sqrt{3}\right)^n + \left(3 - \sqrt{3}\right)\left(2 - \sqrt{3}\right)^n\right)$, $c_{2n+1} = 0$.

$$c_n = \begin{cases} \frac{1}{6}\left(\left(3 + \sqrt{3}\right)\left(2 + \sqrt{3}\right)^{\frac{n}{2}} + \left(3 - \sqrt{3}\right)\left(2 - \sqrt{3}\right)^{\frac{n}{2}}\right) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Special Problem 4

Let D be a convex polygon with $n+2$ sides, such that no three diagonals intersect at a common point. Let d_n be the number of regions that the interior of D is divided by the diagonals into.

(a) Derive an explicit formula for the generating function $A(x)=\sum_{n\geq 1} d_n x^n$.

(b) Derive an explicit formula for d_n.

Answer:

(b) Let v_n be the number of intersection points of diagonals.

$$\begin{aligned} v_n &= \frac{(n+2)\sum_k k(n-k)}{4} \\ &= \frac{(n+2)}{4}\left(n\sum_k k-\sum_k k^2\right) \\ &= \frac{(n+2)}{4}\left(n\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{(n-1)n(n+1)(n+2)}{24} \end{aligned}$$

Let e_n be the number of segments that the diagonals are divided into.

$$e_n=\frac{(n-1)(n+2)}{2}+2v_n$$

In Euler's Formula, $V=v_n+n+2$, $E=e_n+n+2$, $D=d_n+1$, $V-E+D=2$.
$d_n=1+v_n+\frac{(n-1)(n+2)}{2}=1+\frac{(n-1)n(n+1)(n+2)}{24}+\frac{(n-1)(n+2)}{2}$.
(a)

$$\begin{aligned} A(x) &= \sum_n d_n x^n \\ &= \sum_n x^n\left(1+\frac{(n-1)n(n+1)(n+2)}{24}+\frac{(n-1)(n+2)}{2}\right) \\ &= C_0+\frac{C_1}{1-x}+\frac{C_2}{(1-x)^2}+\frac{C_3}{(1-x)^3}+\frac{C_4}{(1-x)^4}+\frac{C_5}{(1-x)^5} \end{aligned}$$

Unfold and compare coefficients, we can get the value of C_i, and then an explicit formula for $A(x)$.

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