

# Mathematics for Computer Science: Homework 7

Instructed by *Andrew C. Yao & Wang Yuexuan*

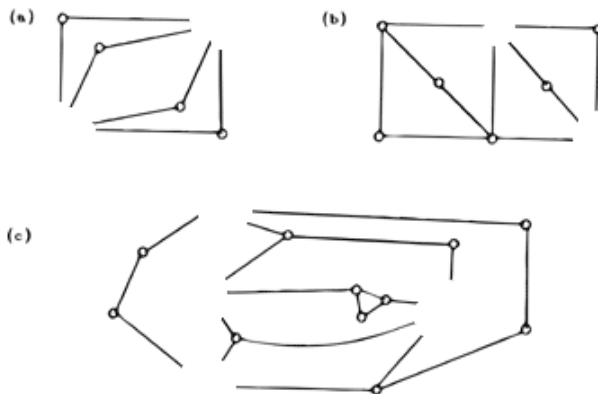
Due on April 22, 2010

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## Exercise 2

Prove that there is no Hamiltonian circuit in each of the following graphs.

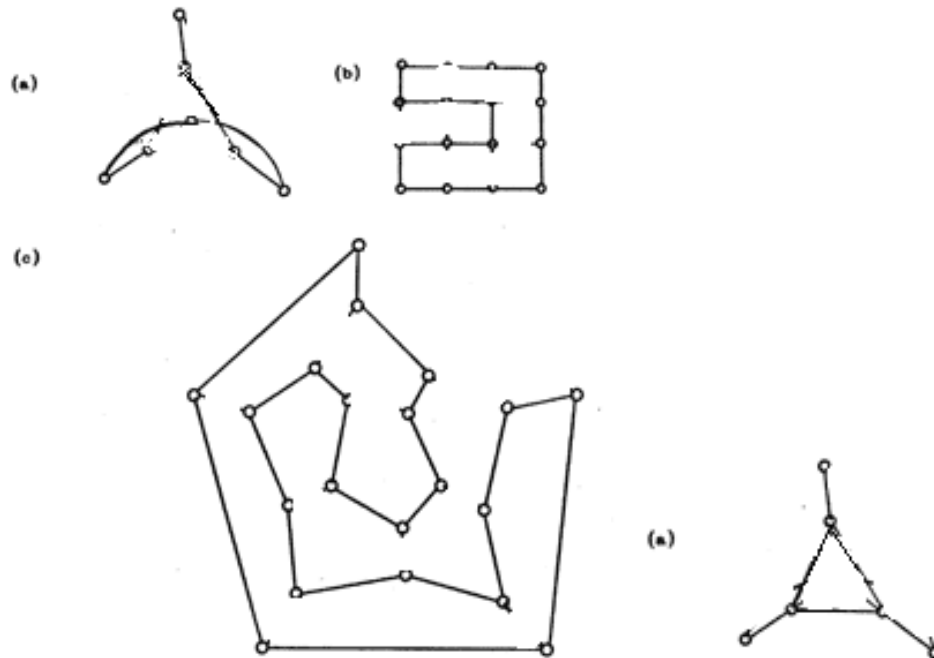
**Answer:** Cut off  $n$  nodes, if it turns to  $n + 1$  disconnected component, there's no Hamiltonian circuit.



## Exercise 4

Which of the following graphs have Hamiltonian circuits, and which have only Hamiltonian paths?

**Answer:**



(a) only path, no circuits.

Proof of no circuits: Cut off the centre node, there's no Hamiltonian paths left. (For there's 3 nodes with only one degree.)

(b) circuits.

(c) circuits.

## Exercise 5

Six points in general position are given in three-dimensional space (that is, no 3 are collinear and no 4 are coplanar). The  $C(6, 2) = 15$  segments joining them in pairs are colored individually either red or blue at random. Prove that some triangle has all its sides the same color.

**Answer:**

Pick a point  $A_1$ , there is at least 3 segments in  $\{A_1A_k\}$  which is in the same color. Assume they're  $A_1A_2$ ,  $A_1A_3$ ,  $A_1A_4$ , and all in red. If  $A_2A_3$ ,  $A_2A_4$ ,  $A_3A_4$  is all blue, a triangle  $A_2A_3A_4$  is found. Otherwise we can assume  $A_2A_3$  is red, thus  $A_1A_2A_3$  is a red triangle.

## Exercise 6

A set of moves in chess which takes a knight successively through all 64 squares is called a knight's tour. If the knight can go from the last square to the first one in one move, and thus go all around again, the tour is called re-entrant. A re-entrant knight's tour corresponds to a Hamiltonian circuit in a graph which has a vertex for each square and an edge joining the vertices representing squares  $X$  and  $Y$  if and only if a knight can go from  $X$  to  $Y$  in one move. Show that there is no re-entrant knight's tour on any chessboard which has dimensions 4 by  $n$ ,  $n$  a natural number.

**Answer:**

Color the chessboard like this:

A	<b>B</b>	A	B	...
B	A	B	A	...
A	B	A	B	...
B	A	B	A	...

Each move is from A to B or from B to A. Starting from the top-left A, it will be  $k_1$ -th move going to **B**, where  $k_1$  is odd.

A	<b>A</b>	A	...	A
B	B	B	...	B
B	B	B	...	B
A	A	A	...	A

Each move is from A to B or from B to A or from B to B. But if once we go from B to B, there will be more B ( $\frac{AB+BA+2BB}{2}$  times in Hamiltonian circuit) than A ( $\frac{AB+BA}{2}$ ). That's impossible. So we cannot go from B to B. Starting from the top-left A, it will be  $k_2$ -th move going to **A**, where  $k_2$  is even.

However, **A** and **B** are the same point, and cannot be visited twice. So  $k_1 = k_2$ . That's a contradiction, for which there is no re-entrant knight's tour on such chessboard.

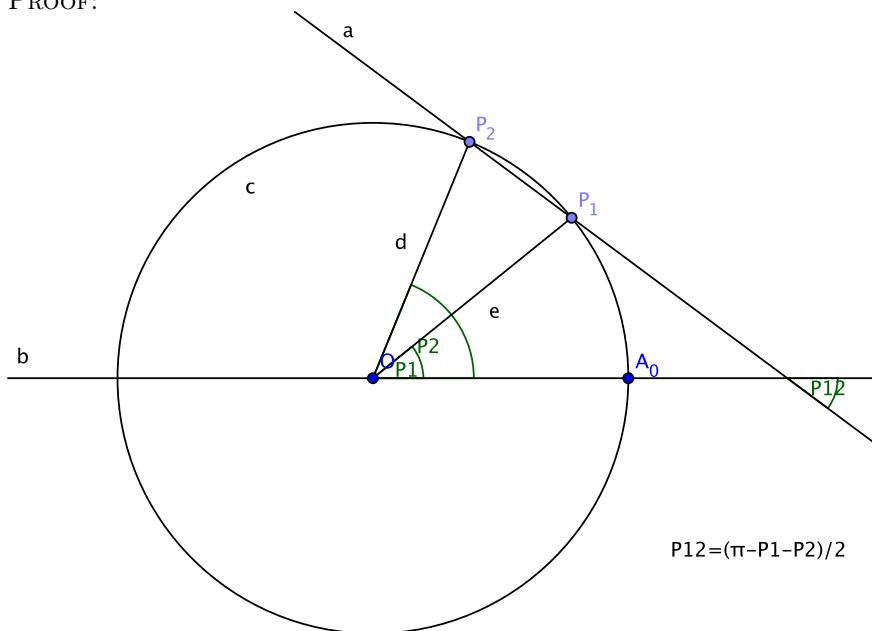
### Exercise 10

Let  $A_0, A_1, A_2, \dots, A_{2n-1}$  denote, in cyclic order, the vertices of a regular  $2n$ -gon. Let all the sides and diagonals be drawn to give graph  $G$ . Prove that every Hamiltonian circuit of  $G$  must contain two edges which are parallel lines in the diagram.

**Answer:**

LEMMA 1: Two edges  $(A_{p_1}, A_{p_2})$  and  $(A_{q_1}, A_{q_2})$  are parallel lines.  $\Leftrightarrow p_1 + p_2 \equiv q_1 + q_2 \pmod{2n}$ .

PROOF:



Assume there are Hamiltonian circuit  $A_{P(0)}A_{P(1)} \dots A_{P(2n-1)}$  without parallel lines, then let

$$\begin{cases} X = \{P(0) + P(1) \pmod{2n}, P(1) + P(2) \pmod{2n}, \dots, P(2n-1) + P(0) \pmod{2n}\} \\ Y = \{0, 1, \dots, 2n-1\} \end{cases}$$

$X \subset Y, |X| = |Y| \Rightarrow X = Y \Rightarrow \sum_{x \in X} x \equiv \sum_{y \in Y} y \Rightarrow 2S \equiv S \pmod{2n}$ , in which  $S = \sum_{y \in Y} y = n(2n-1) \Rightarrow 2n|2S - S \Rightarrow 2n|n(2n-1) \Rightarrow 2|2n-1$ , That leads to a contradiction, for which every Hamiltonian circuit of  $G$  must contain two edges which are parallel lines in the diagram.

## Special Problem

Consider the series expansion of the cotangent function  $\cot(x) = \frac{1}{x} + \sum_{n \geq 0} a^n x_n$ . Use Cauchy's Residue Theorem to derive an expression for  $a_n$ .

### Answer:

Let  $C$  be a circle centered at original point with radius  $R(C)$ . Use Cauchy's Residue Theorem, we have

$$\begin{aligned} a_n + \sum_k \operatorname{Res} \left( \frac{\cot z - z^{-1}}{z^{n+1}}, k\pi \right) &= \frac{1}{2\pi i} \oint_C \frac{\cot z - z^{-1}}{z^{n+1}} dz \\ a_n + \sum_{|k\pi| < R(C)} \operatorname{Res} \left( \frac{\cot z - z^{-1}}{z^{n+1}}, k\pi \right) &= \lim_{R(C) \rightarrow \infty} \frac{1}{2\pi i} \oint_C \frac{\cot z - z^{-1}}{z^{n+1}} dz \\ a_n + 2 \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n+1}} &= 0 \\ a_n &= -2 \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n+1}} \end{aligned}$$

, in which  $\frac{\cot z - z^{-1}}{z^{n+1}} = \frac{\cos z - z^{-1} \sin z}{z^{n+1} \sin z}$ ,  $\sin z|_{z=k\pi} = 0$ ,  $(\sin z)'|_{z=k\pi} \neq 0$ , so  $\operatorname{Res} \left( \frac{\cot z - z^{-1}}{z^{n+1}}, k\pi \right) = \lim_{z \rightarrow k\pi} \frac{\cot z - z^{-1}}{z^{n+1}} = \frac{1}{(k\pi)^{n+1}}$ .

For  $\lim_{R(C) \rightarrow \infty} \frac{1}{2\pi i} \oint_C \frac{\cot z - z^{-1}}{z^{n+1}} dz$  which is used in the equation above, we have

$$\begin{aligned} \left| \frac{\cot z - z^{-1}}{z^{n+1}} \right| &= \left| \frac{\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} i - z^{-1}}{z^{n+1}} \right| \\ &\leq \left| z^{-n-1} \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right| + |z^{-n-2}| \\ &= |z|^{-n-1} \left| 1 + \frac{1 + e^{-2iz}}{1 - e^{-2iz}} \right| + |z|^{-n-2} \\ &\leq 2 \frac{|z|^{-n-1}}{|e^{2iz} - 1|} + |z|^{-n-1} + |z|^{-n-2} \rightarrow 0 \end{aligned}$$

, when  $R \rightarrow \infty$ .

**Acknowledgement:** Answers here are all original, except for the special problem, which is discussed with Mao Jiaxin, J92.