# Mathematics for Computer Science: Homework 9

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## LPV 8.5.9

Prove that in every tree, any two paths with maximum length have a node in common. This is not true if we consider two maximal paths.

**Answer:** Suppose there're two paths with maximum length and no common nodes, which is $A_1A_2\dots A_t$ and $B_1B_2\dots B_t$, $A_i \neq B_j$. There is a path from $A_1$to $B_1$ in the tree, and let it be $A_1A_2\dots A_pB_qB_{q-1}\dots B_1$, $1 \le p \le t$, $1 \le q \le t$, with length $p+q \le t$. So $A_pB_q$ is a edge in the tree, and $A_tA_{t-1}\dots A_pB_qB_{q+1}\dots B_t$ is a path, too, with length $2t+2-p-q \le t$. Thus $2t+2 \le 2t$, which is impossible.

## LPV 8.5.10

If $C$ is a cycle, and $e$ is an edge connecting two nonadjacent nodes of $C$, then we call $e$ a chord of $C$. Prove that if every node of a graph $G$ has degree at least 3, then $G$ contains a cycle with a chord.

**Answer:** Take a longest path $A_1A_2\dots A_n$. Since $A_1$has degree at least 3, $A_1$ must be connected with $A_j$, $A_k$ where $2 < j < k \le n$, or there will be a longer path. Thus $A_1A_2\dots A_k$ is a cycle with a chord $A_1A_j$.

## LPV 8.5.11

**Answer:** Let the circle be $A_0A_1\dots A_{n-1}$, $A_n = A_0$, and the added edge be $A_0A_2$. Deleting any two edges from this graph leads to a tree, except for $(A_0A_1, A_1A_2)$ or $(A_iA_{i+1}, A_jA_{j+1})$ with $2 \le i \neq j \le n-1$. Thus the answer is $\binom{n+1}{2} - \binom{n-2}{2} - 1 = 3n-4$.

## LPV 9.2.4

Describe how you can find a spanning tree for which:
(a) the product of the edge-costs is minimal.
(b) the maximum of the edge-costs is minimal.

**Answer:**
(a) Change the cost of an edge $c$ to $\ln c$ before using the Kruskal's algorithm. However, it doesn't change the order of edges. In fact, we can directly use Kruskal's algorithm without any change.

(b) Let $M$ be a large number. Change the cost of an edge $c$ to $M^c$ before using the Kruskal's Algorithm. However, it doesn't change the order of edges. In fact, we can directly use Kruskal's algorithm without any change.

## LPV 9.2.8

Show by an example that if we don't assume the triangle inequality, then the tour found by the Tree Shortcut Algorithm can be longer than 1000 times the optimum tour.

**Answer:** Four points: $A$, $B$, $C$, $D$, with $AB = 1$, $BC = 1$, $CD = 1$, $DA = 2$, $AC = 2500$, $BD = 2500$. Optimum tour: $A \to B \to C \to D \to A = 1 + 1 + 1 + 2 = 5$. MST:$\{AB, BC, CD\}$. A possible tree tour: $B \to A \to B \to C \to D \to C \to B$. Shortcut: $B \to A \to C \to D \to B = 1 + 2500 + 1 + 2500 = 5002 > 5 \times 1000$.

## Special Problem 1

For any integer $n > 0$, let $G_n = (V, E)$ be a graph on $2n$ vertices, where $V = \{1, 2, 3, \ldots, 2n\}$, and $E = \{\{i, j\} \,|1 \leq i < j \leq 2n, j \neq n + i\}$. Answer the following questions, each with a concise but rigorous justification.

(a) For what values of $n$ are $G_n$ Eulerian?
(b) For what values of $n$ do $G_n$ contain a Hamiltonian circuit?
(c) What is $\omega(G_n)$, the size of the largest clique in $G_n$?
(d) What is $\chi(G_n)$, the chromatic number of $G_n$?

**Answer:**

(a) For any $n > 1$, $G_n$ is connected. For any $n$, $d(V_i) = 2n - 2$ is even. So there is a Euler circuit in $G_n$ for any $n > 1$.

(b) For any $n > 1$, $V_1 V_2 \ldots V_{2n} V_1$ is a Hamiltonian circuit. For $n = 1$, $G_n$ is not connected.

(c) $\omega(G_n) = n$. On one hand, if there's a clique with size $n+1$, there must be two of its points in the same set in $\{\{1, n + 1\}, \{2, n + 2\}, \ldots \{n, 2n\}\}$. But there's no edge $(i, n + i)$. On the other hand, $V_1 V_2 \ldots V_n$ is a clique with size $n$.

(d) $\chi(G_n) = n$. On one hand, $\chi(G_n) \geq \omega(G_n) = n$. On the other hand, we can color $\{i, n + i\}$ with color $i$, using exactly $n$ colors, so that no two adjacent vertices share the same color.

## Special Problem 2

Let $n = 3$, and $D = G_n$ as defined in Problem 1.
(a) What is the Laplacian for the graph D?
(b) Determine the number of spanning trees in D.

**Answer:**

(a) $$\mathrm{L}(D) = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ -1 & -1 & 0 & -1 & -1 & 4 \end{pmatrix}.$$

Special Problem 2 continued on next page. . . 

(b) $\begin{vmatrix} 4 & -1 & -1 & 0 & -1 \\ -1 & 4 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & 4 \end{vmatrix}$.

# Special Problem 3

Let $G = (V, E)$ be a graph where $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_{n-1}\}$. Let $A = (a_{i,j})$ be the $n \times (n-1)$ vertex-edge incidence matrix as defined in class. That is, for each $e_k = \{i, j\}$ where $i < j$, we have $a_{i,k} = 1$ and $a_{j,k} = -1$, and all other entries of $A$ are 0. Let $A'$ be the square matrix resulted if we delete from $A$ its last row. Prove $|\det(A')| = 1$ if $G$ is a tree, and 0 otherwise.

**Answer:**

**Lemma.1** $|\det(A')| = 0$ or $|\det(A')| = 1$.

**Proof** There're at most 2 non-zero elements in a column. If there is a column with all zero, $\det(A') = 0$. Else if all columns have two non-zero elements, sum of all rows will be all zero, thus $\mathrm{r}(A') < n-1$, $\det(A') = 0$. Else there's a column with exactly one non-zero element. Expand $A'$ along this column, $\det(A') = \pm\det(A'_1)$, and by induction we get $\det(A') = 0$ or $\det(A') = \pm 1$.

**Lemma.2** $|\det(A')| = 0$ if $G$ is not a tree.

**Proof** Obviously, there's a loop with length at most $n-2$ in $G$. Sum all columns which is an edge in this loop in $A$, we get $\vec{0}$ (for each point, one edge in with 1, one edge out with $-1$). So $\mathrm{r}(A) \leq n-2$, $\mathrm{r}(A') \leq \mathrm{r}(A) < n-1$, $\Rightarrow |\det(A')| = 0$.

**Lemma.3** $G$ is not a tree if $|\det(A')| = 0$.

**Proof** Choose the smallest set of linearly dependent rows. If there is a column with exactly one non-zero element in these rows, we can delete the element's row to get a smaller set since other rows in the set can't represent this row linearly. That's impossible since we have chosen the smallest set. So there's no column with elements both in and out of this set of rows. In another word, there's no edge connect the point set of these rows to other points. Thus the graph is not connected, and it's not a tree.

If $G$ is a tree, by Lemma.3, we get $\det(A') \neq 0$, by Lemma.1, $|\det(A')| = 1$. If not, by Lemma.2, $|\det(A')| = 0$.

**Acknowledgement:** Answers here are all original.